THE APPLICATION OF CIRCULAR STATISTICS TO PSYCHOPHYSICAL RESEARCH

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Abstract

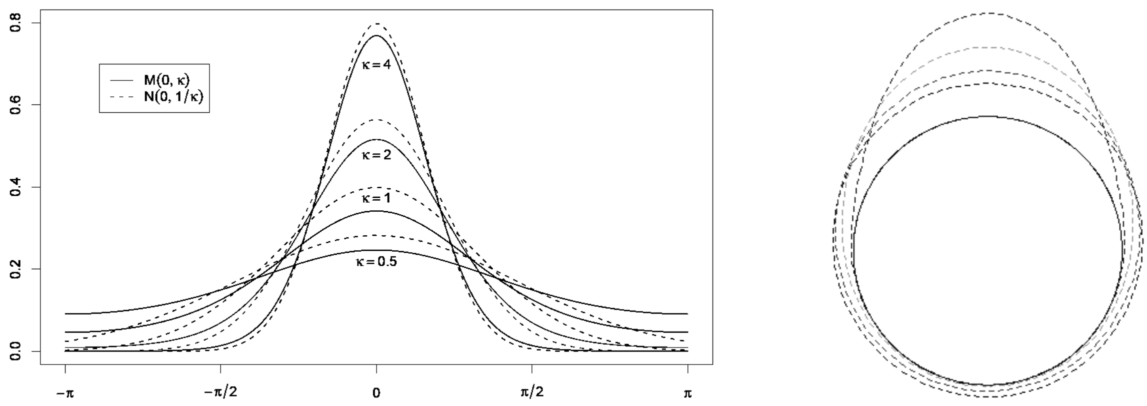
Directional data arising from psychophysical studies requires careful processing due to its cyclical nature. Unlike linear variables such as response time and intensity, directions can be represented as angles or vectors on a circle, may have no natural zero-point or magnitude, and are defined on a periodic circle rather than an infinite line. Because of these unique features, directional data necessitates the use of circular statistical methods; however, unfamiliarity and limited availability of software that support circular data analysis have largely led to its under-use and misapplication. Here the implications of using linear statistics for circular data were explored by submitting data from a behavioural study to both circular and linear statistical analysis.

In many areas of psychophysical research, measurements are in the form of angles or directions as a unit vector. For instance, in an auditory localization task, the perceived direction of a sound can be measured in degrees, or in a colour-matching task, responses can be indicated on a colour wheel as an angle. Directional measures can also include compass direction, and time cycles such as the 24-hour clock, week or calendar year. One of the pioneers of circular statistics was Florence Nightingale who had invented the polar-area diagram to illustrate the incidences of preventable causes of mortality over the months of the Crimean War. In fact, the polar-area diagram was so compelling it persuaded Queen Victoria to reform hospital sanitation practices and revolutionized the field of nursing (Rehmeyer, 2008; Lipsey, 1993).

For several theoretical and empirical reasons, using linear methods to assess circular data can result in absurd interpretations due to the arbitrary choice of origin—the 0° point on the circle—and the ability to measure the data in a bi-directional manner (Zar, 1999). For instance, an observer's response of 135°and a correct angle of 45°, can be interpreted in the clockwise direction as a 90° error, or in the counterclockwise direction as a 270° error. Or in an extreme case, one might take a completely naive approach to analysis and simply plug in the dependent variable as is or use difference values as a measure of error, as opposed to a more reasonable approach of using absolute differences.

Another point of distinction is that the circular probability distribution has a total probability that is on the circumference of a unit circle, and therefore the normal distribution is not a satisfactory probability model (Jammalamadaka & SenGupta, 2001, p. 25). One type of circular distribution is the von Mises distribution, which can be thought of as analogue to the normal distribution with a dispersion that is quantified by the parameter kappa, κ, with uniformity indicated by $\kappa = 0$ (Fisher, 1993). Figure 1 shows a linear and polar representation of the von Mises distribution; density on the polar plot is the distance from the circumference of the circle.

Clearly, there are several unique features that make circular data considerably different from "linear" values. Although angular direction is often used as a measure in psychophysical research, circular statistical methods are not always employed when it is appropriate to do so. In order to explore the implications of submitting circular data to a linear analysis, comparable circular and linear statistical approaches were used to examine circular data on visual spatial memory. The linear approach involved the previously mentioned naive and absolute difference methods.



(a) von Mises and normal densities

(b) Probability density functions, M(0, κ)

Fig. 1: Linear 1(a) and circular 1(b) presentations of the von Mises densities with μ = 0° and κ = 0.5, 1, 2, and 4.

Methods

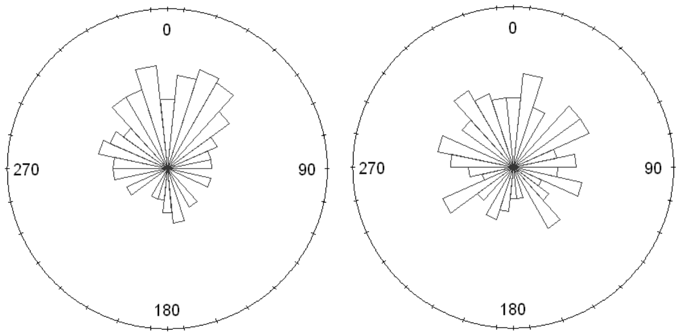
The dataset was obtained from a study by (Guterman, Allison, Jennings, Craig, Gauthier, & Macuda (2009) who examined the effects of Field of View (FoV) restriction on spatial memory. Subjects learned a novel environment while wearing goggles that restricted their FoV to 40° or 90°. They were then asked to indicate the direction of a test object relative to a reference object. The reference and test object in question were both search targets, distracters, or a mix of targets and distracters; this paper discusses the former two conditions. All of the circular analyses were performed using R 2.8.0 (Lund & Agostinelli, 2008) with the "Circular" package (http://cran.r-project.org/web/packages/circular/index.html).

Results

Circular statistics approach

Circular-circular regressions were performed between the judged angles as the dependent variable, and the correct angles as the independent variable. These models were "circular-circular" as both the dependent and independent variable were angular and hence had periodicity of 360°. The regression residuals were used as a convenient measure of (fitting) error and precision of participants' angle judgments. The data were grouped by FoV (40° and 90°) and item type. Software that supported the more suitable approach of multiple circular-circular regression was not available, so simple circular-circular regressions were performed.

Screening procedures. Regression diagnostics were based on the circular residuals. First, circular normality was assessed using Watson's one-sample U^2 goodness of fit test. This statistic tests the null hypothesis that the residuals conform to the von Mises distribution. Results of Watson's U^2 test showed that all of the regression models were von Mises, U^2 ranged from 0.0176 to 0.0666, with $p > 0.10$ for all tests. Next, the dataset was examined for outliers by visually inspecting the probability-probability (P-P) plots for points deviating from the fit line y = x, as recommended by (Jammalamadaka & SenGupta, 2001). Points that were far away from the 45° reference line could be suspected as possible outliers. Inspection of the plots did not reveal any aberrant points.



(a) Targets (b) Distracters

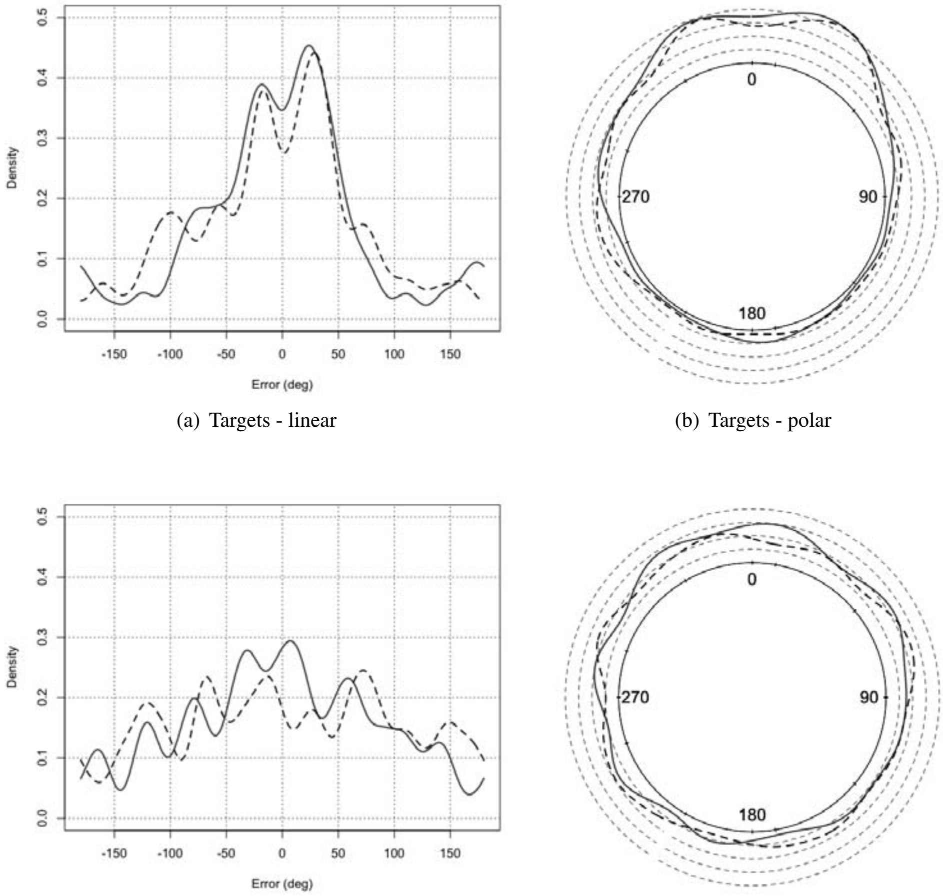
Fig. 2: Rose diagrams of the 40° FoV group errors.

Circular histograms called *rose diagrams* were created to explore the variables FoV and item type. Figure 2 shows rose diagrams of the frequency of errors (i.e., residuals) as a function of angle, for participants in the 40° FoV group. With the possible errors ranging from 0° to 180°, we may consider 90° to be the chance level. In the Targets condition (see Figure 2a), judgment errors can be seen as being concentrated around 0°. Although the angular errors were quite broad, on average they were less than 90°, indicating that they could generally perform the task. In Figure 2(b), the errors were more uniformly spread on the circle and less concentrated around 0°, and this suggests that responses were less precise when the two objects in the question were not search targets. A similar pattern was found in the rose diagrams of the 90° FoV group.

Since task performance was a function of participants' estimates and the correct angles, the interdependence of these variables was explored. Circular-circular correlations were computed to discern the strength of the bivariate relationship between the judged and correct angles. This measurement is a circular version of Pearson's correlation with the key difference being that (x_i - $\bar{x}$) and (y_i - $\bar{y}$) become sin(x_i - $\bar{x}$) and sin(y_i - $\bar{y}$), and the sample correlations are denoted by r_c. The derivation of r_c involves both the sine and cosine, but due to the theoretical dependence of these two trigonometric functions on each other, the formula can be simplified to involve only the sine (Fisher, 1993). Also, the sign of a circular correlation—though it is generally not interpreted due to the complexity of its interpretation—provides information about the rotational and reflectional dependencies of the variables. For instance, a rotation may suggest a consistent directional bias, whereas a reflection might indicate a reflection about the observer's axis of reference.

In the Targets condition, the angle vectors for the 90° FoV group showed a weak but significant negative correlation, r_c = -0.21, *p* = 0.04. This suggests that the two angle vectors have a minor reflectional relationship, which may appear graphically as a reflection about a polar axis. In the Distracters condition, the angles for the 40° FoV showed a weak but significant positive correlation, r_c = 0.30, *p* = 0.0044, indicating that the association between the variables is partially explained by the rotational model. In both of these instances, we see a consistent relationship between the estimated and observed angles. Still, the weak correlations suggest that participants' estimates lacked precision. Further analysis will help put these results into perspective with respect to task performance.

A test of estimate precision is Kuiper's one-sample *V* test of uniformity on the circle. Good performance should be reflected by estimate errors concentrated around 0°, whereas random unbiased guessing would result in errors uniformly distributed on the circle. Interpretation of the *V* test was facilitated by assessing plots of the density (or frequency) of errors as a function of angle. Figure 3 shows circular errors plotted on both linear and polar plots. The graphs



(a) Targets - linear

(b) Targets - polar

(c) Distracters - linear

(d) Distracters - polar

Fig. 3: Linear (left) and polar (right) plots of the errors as a function of angle (Solid line = 40°, dashed line = 90°). Dotted lines in the polar plots represent densities of 0.1, 0.2, 0.3, and 0.4.

show that errors were mostly concentrated in the Targets condition, but more wide-spread or uniform in the Distracters condition. Presenting these plots side-by-side also demonstrates how linear and polar plots can appear very different.

Uniformity of the errors was confirmed for the Distracters condition, but only for the 90° FoV group (V_{40° = 2.183, $p < .01$; V_{90° = 1.500, $p > .15$). In contrast, the distribution of errors for the Targets condition, for both FoV groups, was not uniform, indicating that they were not responding at random and could effectively perform the task (V_{40° = 4.164, $p < .01$; V_{90° = 3.330, $p < .01$). Figure 3 shows these patterns and the unexpected poorer performance by the 90° FoV group in the distracters condition. It is possible that the wider FoV group remembered fewer distracter items as they may have foveated less of the scene than their counterparts.

Circular-circular regressions. Based on converging lines of evidence that the responses for the Distracters condition were essentially no better than random, these conditions were dismissed from the formal circular-circular regression analysis. In other words, it made little sense to ask, "Which of the two FoV groups had better guessers?" Hence, the following presents the results of the 40° and 90° FoV group for the Targets condition.

A circular-circular regression model was fitted to the data for the 40° and 90° FoV groups. The regression parameters were marginally significant for the 1^{st} order terms for the cosine and sine for the 40° FoV condition (p_{40° = 0.0283 and 0.0712) and significant for the 90° FoV condition ($p_{90^\circ} < 0.001$ and 0.001), and higher order trigonometric models were not significant at the 0.05/2 = 0.025 level. We can think of these p-values as indicating the significance of the relation of the cosine and sine of the estimated angle (y) with the actual angle (x), respectively. Thus, the regression models were a reasonable fit and the estimated angles followed the actual angles. The coefficients of the fitted model predicting the cosine and sine of y are shown in the circular regression equations (see Eq. 1 and 2).

$$\text{FoV} = 40^\circ \quad \cos(y) = 0.602\cdot\cos(x) - 0.090\cdot\sin(x) - 0.018 \qquad (1)$$
$$\sin(y) = -0.080\cdot\cos(x) + 0.414\cdot\sin(x) - 0.144$$

$$\text{FoV} = 90^\circ \quad \cos(y) = 0.484\cdot\cos(x) - 0.099\cdot\sin(x) - 0.046 \qquad (2)$$
$$\sin(y) = -0.090\cdot\cos(x) + 0.462\cdot\sin(x) - 0.137$$

The equal kappa test for homogeneity of concentration parameters was used to determine if estimate precision differed between the 40° and 90° FoV groups. A close relation between the estimated angles and predicted angles, as indicated by the residuals, would result in a large κ value. The results showed that κ for the groups were not significantly different, $\chi^2 = 1.77$, df = 1, $p = .2782$ ($\kappa_{40^\circ} = 1.30$ and $\kappa_{90^\circ} = 1.02$). This closeness is apparent in Figure 3 which shows similar density on the circle for the 40° and 90° group. Thus, the circular analysis indicated that FoV did not impact performance when both the reference and test objects were previous search targets.

Naive approach

The methodology paralleled the circular approach except that simple linear regressions were performed on the estimated and actual angles. The analysis focused on the Targets condition based on the earlier finding that responses in the Distracters condition were no better than random. Performance of the 40° and 90° FoV groups were compared using Levene's test of equality of variance as an alternative to the equal kappa test to assess dispersion. The R^2 for the regressions were not significant at the 0.05/2 = 0.025 level, $R^2_{40^\circ} = 0.02$, $F = 2.38$, $p = 0.126$, $B = 0.14$ and $R^2_{90^\circ} = 0.03$, $F = 3.42$, $p = 0.067$, $B = 0.17$. Thus, the regression models were not a good fit and the estimated angles did not follow the actual angles. Results of Levene's test showed that the FoV groups were not significantly different, $F(1, 206) = 0.32$, $p = 0.5729$. Unlike the circular statistics approach, the estimated angles could not be used to satisfactorily predict the actual angles.

Absolute differences approach

For this model, we attempted to linearize the response variable by taking the absolute difference of the estimate and actual angles. This transformation made it possible to perform a standard multiple regression to examine the relationship among the participants' estimate errors, and the FoV and item type conditions. Included in the model was an interaction term to test the hypothesis that the affect of FoV depends on item type, as this was indicated in the circular uniformity tests. Multiple regression was again the more suitable method and, of course, the software is readily available in the linear case. Results of the evaluation of assumptions led to a square-root transformation of the absolute errors to reduce skewness and improve the normality and homoscedasticity of residuals. The multiple R^2 for the regression was significantly different from zero, $F(3, 620) = 17.65$, $p < 0.001$ with R^2 at 0.08. FoV was not a significant predictor of participants' estimate errors, $B = 1.22$, $t = 1.61$ with $p = 0.108$. Also, the interaction between FoV and item type was not significant, $B = -0.47$, $t = -1.33$ with $p = 0.185$. However, the item type was a significant predictor of estimate error, $B = 1.95$, $t = 3.50$ with $p < 0.001$. As seen in Figure 4, participants showed smaller errors when both the test and reference object were search targets, and larger errors when both of the objects were distracters in the environment.

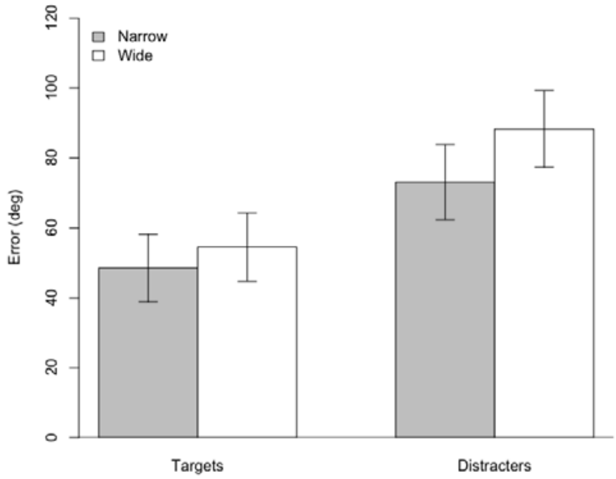


Fig. 4: Absolute differences approach.

Discussion

Circular techniques allowed for a richer understanding of the relationship among the variables, such as the circular direction and uniformity of the errors, which were visually more accessible to interpretation on circular plots. Furthermore, when the circular data was linearized to absolute differences, a satisfactory fit to the normal distribution was difficult to achieve; this may suggest that linearizing the errors in this fashion does not completely rid the data of its circular tendencies. It is possible that a different dataset could have yielded different findings. Also, using the naive approach complicated interpretation of the data and the findings were in contrast to the other methods. Thus, in general, it is recommended that circular data be statistically treated using the appropriate circular statistical method.

Finally, software was not available for doing a multiple regression of a circular response variable versus circular and linear predictor variables. Fourier analysis methods offer very similar methods of regression and may be applicable in attempts to tease out variables. Instead, six individual circular-circular regressions and additional statistical tests were performed which were a fair alternative and interim approach. The somewhat involved theoretical derivations and software need to be developed to encourage and facilitate the use of these methods, and to enrich understanding of circular variables in psychophysical and other experimental paradigms.

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