FROM MATHEMATICS TO PSYCHOPHYSICS: DAVID HILBERT AND THE "FECHNER CASE"

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Abstract

David Hilbert (1862-1943), the great mathematician from Königsberg and Göttingen, is a relevant figure in the science of the nineteenth and twentieth centuries, who marked the development of not only mathematical but also physical activities. We present here a sketch of Hilbert's figure and work, in particular of his contribution to the debate which ensued after the publication of Elemente *and the program of psychophysics by Fechner. In his lecture course* Logische Principien des mathematischen Denkens*, held at the University of Göttingen in 1905, Hilbert proposed an axiomatization of psychophysics. This formulation may be seen as an interesting case of Hilbert's famous axiomatic approach in natural science.*

Introduction

Fechner's work immediately had great impact on the scientific community: For the first time, with *Elemente der Psychophysik* (1860), a rigorous project of empirical and experimental research, founded and guaranteed by the possibility of measuring phenomena, was started in psychology. Since its publication, *Elemente der Psychophysik* has provoked, as is understandable considering the importance of such a project, a wide and lively debate among scholars (see Brožek & Gundlach, 1988; Heidelberger, 1993; Murray, 1993; Zudini, 2009, 2011). The model proposed in *Elemente* became that of reference: a model to criticize, correct, or confute, in the methodological aspects of its (empirical and mathematical) procedures or even in its psychophysical, physiological, or, in a strict sense, psychological value itself; in certain cases, it was a model to reject in a radical way, on the basis of the assumption that it was impossible to measure sensations and, in general, psychical magnitudes and therefore to make a scientific study on them.

The debate about Fechner's work was initially developed around three types of questions: The first concerned how correct the derivation of the law proposed by Fechner was, starting from the experimental data and from the mathematical tools he used; the second related to the very nature of the law. A third type of problem dealt with the possibility of measuring sensations and psychical magnitudes generally. Many scholars, from very different disciplines, studied the "Fechner case" and the discipline of psychophysics. Among them, there were the greatest figures of the science of the nineteenth and twentieth centuries, such as Hermann von Helmholtz, Ewald Hering, Ernst Mach, Wilhelm Wundt, Franz Brentano, Joseph Antoine F. Plateau, Joseph Delboeuf, and David Hilbert.

Hilbert, a great mathematician

David Hilbert (1862-1943), born in Königsberg (now Kaliningrad), birthplace of Immanuel Kant and known in mathematics for the "bridge problem", was professor first at the University of Königsberg itself, then, from 1895, at the University of Göttingen, where he taught until the end of his career. Hilbert was one of the greatest mathematicians at the turn of the nineteenth and twentieth centuries, whose name is now linked with the concept of "Hilbert space", crucial in functional analysis.

Hilbert was important for significant contributions in several areas of mathematics and physics, from invariant theory, to algebraic number fields, calculus of variations, integral equations, mathematical physics, logic and theory of demonstration, as well as foundations of geometry and mathematics generally. In particular, Hilbert was the founder of "meta-mathematics" (i.e. the study of the accuracy of the methods used in mathematics); he was the head of the so-called "formalist school" – the school of thought which saw mathematics as a set of formal systems. Opposed to this was the "intuitionist school", founded by the Dutch mathematician and philosopher Luitzen Egbertus Jan Brouwer (1881-1966), traditionally referred to as "L. E. J. Brouwer", according to whom language and logic are not the presuppositions of mathematics, which instead has its origin in intuition that makes its concepts and deductions immediately clear.

Hilbert's enthusiasm for mathematics and his confidence in its vitality and potential are evident in his famous address "Mathematical Problems" delivered to the Second International Congress of Mathematicians held in Paris in 1900, a great opportunity to review the issues and problems of science pending at the turn of the century. Hilbert's optimistic contribution, containing a list of 23 problems, a veritable research program for the "coming generations", i.e. for the mathematicians of the new century, has played a determining role in marking the development of not only mathematical but also physical activities. According to Hilbert, the fundamental dynamics of the development of mathematics and, at the same time, the impetus to the process of mathematization of the other sciences lay in the continuous interaction between the free creations of reason and the knowledge of the phenomena of the external world. The rigor of the demonstrations, that peculiar quality of mathematics understood by Hilbert as "a general philosophical need of our reason", proved to be necessary in dealing with both issues of analysis and those that originated in the external world, the world of empirical experience.

The axiomatic approach in mathematics

Hilbert had given a "taste" of his approach already in his studies on the foundations of geometry, the subject of some of his lecture courses and of the work *Grundlagen der Geometrie*, first published in 1899 and revised several times over the years. *Grundlagen der Geometrie* is fundamental in the evolution of geometrical thought: In it Hilbert proposes to establish for geometry a complete and as simple as possible system of axioms and to deduce from it the most important geometrical propositions, highlighting the significance of the various groups of axioms and the extent of the conclusions which are drawn from them.

Grundlagen is testimony to the new way of conceiving geometry (as a hypothetical-deductive system) which had been emerging in the second half of the nineteenth century and which resulted necessarily in a new way of looking at the definition of the objects of the geometry itself. Hilbert's procedure is different from that of Euclid: It does not give a definition of the objects which are taken into consideration or clarify their nature at the beginning of the treatment; this is done later through the statement of the axioms and not through the names given to the objects themselves or via a reminder of the experience and the external world. Such an attitude is similar to what happens when we invent a game (e.g. cards): The actual nature of the cards (or their value) is specified by the rules of the game which we want to play more than by the figures printed on them; the definition is given implicitly by the statement of the rules and not by a phrase indicating the nature "per genus et differentiam", according to the canons of classical logic. According to this view, the object called "line" by the Euclidean geometry is not the same object which is called "line" by the non-Euclidean geometry, because the axioms of the former geometry are different from those of the latter. It is the axioms that give the rules of the game which we have to use with the objects of the geometry under consideration.

In the usage of undefined concepts (such as point, line, plane, ...) and in the fact that their properties are established only by the axioms, Hilbert follows the German mathematician Moritz Pasch (1843-1930), author of *Vorlesungen über neuere Geometrie* (1882). According to this tradition, it is not necessary to assign any explicit meaning to undefined concepts. These elements (point, line, plane, ...) could then be substituted, as Hilbert said, by tables, chairs, tankards, ... The axioms are not self-evident truths, but implicit definitions of the primitive terms which they contain; they must be considered arbitrary, even if, actually, they are suggested by experience.

After giving an example of an axiomatic construction, Hilbert addresses the problems associated with such a construction (in general, for any axiomatic system), i.e. the consistency and the mutual independence of the axioms. With regard to the consistency, that is, the coherence, he proposes to refer the coherence of the geometry to the coherence of the system of real numbers. Regarding the issue of the independence of the axioms, his discussion is of considerable interest because, when an axiom is proved as independent, at once the legitimate existence of the corresponding "non-…" geometry is suggested (for example, proving the independence of the axiom of parallels is equivalent to proving the possibility of a non-Euclidean geometry). The proof procedure is substantially similar to that applied when proving the compatibility of the axioms, i.e. by using a numerical model.

With his *Grundlagen*, Hilbert played an important role as a mathematical logician. The guarantee of the logical compatibility of the axioms of a certain group is given by Hilbert by building up every time models borrowed from other fields of mathematics. The statement of the logical compatibility of the foundations and the processes of algebra and arithmetic is thus responsible for ensuring the compatibility of the axioms of geometry. This procedure of reference, however, can not be infinite; at some point we must stop and look for a "foothold". Starting from this problem, which concerned above all the foundations of arithmetic and analysis, Hilbert developed the program of creating a "*Beweistheorie*" ("proof theory"), which envisaged the construction of a system of logical procedures that could justify the classical mathematical procedures and were justifiable themselves through finite procedures and above criticism. In other words, the aim was that of constructing a "meta-mathematics", i.e. a meta-theory which had as object mathematics and its methods and was unassailable. This project of creating a "*Beweistheorie*" – which constituted a large part of Hilbert's research program on the foundations of mathematics and was considerably developed – was frustrated by the Austrian mathematician and logician Kurt Gödel (1906-1978), who gave a fatal blow to the attempts of formalist mathematics in 1931 with his famous eponymous theorem. Gödel's contribution showed that in a theory containing arithmetic we could construct a formula whose truth is impossible to be proved and which thus would be an undecidable sentence. This would indicate the incompleteness of the theory, namely its inability to rule on all statements which it could express, regardless of the number of axioms placed at the beginning of the treatment.

The axiomatic approach in natural science and in psychophysics

In Hilbert's conception, the criterion of truth and existence of mathematical objects was then the demonstration of the consistency of the axioms and the theorems derived from them.

From this perspective, every theory was nothing but a kind of frame, a scheme of concepts together with their necessary mutual relationships, applicable to infinite systems of fundamental objects. It was sufficient that the relations between them were established from the axioms for obtaining all the propositions of the theory. The axiomatic method – and this was, according to Hilbert, its essential quality – highlighted the deductive pattern, the dependency link between axioms and theorems.

The research program proposed by Hilbert involved applying the axiomatic method to all branches of physics where mathematics played a dominant role. The aim was clearly stated in Problem 6 presented at the Congress in Paris: mathematical treatment of the axioms of physics, in particular axiomatization of those parts, such as mechanics and the theory of probability, where mathematics was essential. Here was evident the connection with the theories of probabilistic nature on kinetics of gases developed by Rudolf Clausius and Ludwig Boltzmann, as well as with the research on the principles of mechanics conducted by Ernst Mach and Boltzmann himself, in the context of an increasingly marked interest in theoretical physics which Hilbert (with his school) had fostered since the beginning of the twentieth century. In Hilbert's wake contributions were given to the resolution of Problem 6 in thermodynamics by Max Born and Constantin Carathéodory, in quantum mechanics by Hermann Weyl and John von Neumann, in electrodynamics by Hermann Minkowski, in probability theory by Richard von Mises and by Sergei Natanovich Bernstein and Andrey Nikolaevich Kolmogorov in the context of the modern measurement theory.

In Hilbert's conception, all sciences, starting from mechanics, should be treated according to the model set forth in geometry. The lecture course held by Hilbert at the University of Göttingen in 1905 on the axiomatization of physical theories (*Logische Principien des mathematischen Denkens*, 1905; see Corry, 2004) showed how this should be done in practice. A section was dedicated to each of the following disciplines: mechanics, thermodynamics, probability calculus, kinetic theory of gases, insurance mathematics, electrodynamics, and psychophysics. (Other sections, on radiation theory and on relativity, were added in the lecture course held on the same subject in 1913.) Hilbert reviewed the various theories of physics and the different disciplines, with no specific references to the historical background or sources, in order to give a unified view of them to the students. In this perspective, it was assumed that every theory was governed by specific axioms which expressed the mathematical properties establishing relationships among the basic magnitudes pertaining to it. In parallel, there were some general mathematical and variational principles assumed to be valid for all physical theories: Great importance was given to the continuity axiom, for which a general formulation was proposed along with other ones more specific to each theory. The aim was to show how, through these (specific and general) principles, one could derive the basic equations of each theory. The derivation of the equations from the axioms was indicated in a schematic, very rapid way; Hilbert often confined himself to affirming the possibility of such a derivation, considered as plausible and feasible according to the pattern shown for geometry in *Grundlagen der Geometrie*. There was furthermore no general demonstration of the independence, consistency or completeness of the axiomatic systems proposed.

According to this scheme, as mentioned earlier, Hilbert treated psychophysics as the last item after the other disciplines listed. With reference to the work of the Austrian astronomer Egon Ritter von Oppolzer (1869-1907) on the psychological theory of color perception (Oppolzer, 1902-1903), which was based on Fechner's law, the idea was to express the magnitude of brightness (a purely psychological parameter) as a function of the intensity and wavelength (physical parameters).

The axioms assumed to be defined for a collection of "brightnesses" x_1, x_2, …, x_n, as confirmed, according to Hilbert, by experience, were the following (Corry, 2004):

1. *To every pair of brightnesses x_1, x_2, a third one $[x_1,x_2]$ can be associated, called "the brightness of the mixed light of x_1, x_2." Given a second pair of brightnesses x_3, x_4, such that $x_1 = x_3$ and $x_2 = x_4$, then $[x_1,x_2] = [x_3,x_4]$.*

2. *The "mixing" of various brightnesses is associative and commutative.*

3. *By mixing various homogeneous lights of equal wavelengths, the brightness of the mixed light has the same wavelength, while the intensity of the mixed light is the sum of the intensities.*

Let us indicate $[x_1,x_2]$ with $x_{12:}$ We can write x_{12} as a function of the two parameters x_1, x_2, so that

$$x_{12} = f(x_1,x_2).$$

We can then introduce a function F such that

$$F(x_{12}) = F(f(x_1,x_2)) = F(x_1) + F(x_2).$$

From Axiom 3 and from the general postulate of continuity, F, for homogeneous light, is proportional to the intensity. According to Hilbert, the knowledge of this function, which was called "stimulus value", allowed one to establish the whole theory.

Hilbert's work, which is very short, schematic, without reference to the historical background of psychophysics, remains a puzzle from many points of view, particularly concerning Hilbert's relationship with the development of the discipline itself in Germany, and especially in Göttingen, an important driving force of the psychology of the time (see also Corry, 2004).

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