

INSIGHT IN RATIO SCALES

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Abstract

Ratio scales have been a highly controversial issue between physicists and psychologists in the past in regard of the necessity of an empirical addition operation for attaining them. This aspect is very critical, since such an operation is often unavailable in psychophysical experiments. Stevens indicated “equality of ratios” as a distinctive feature for ratio scales, but he did not provide any axiomatization, in support of this statement. In this paper we proposed a reconsideration of ratio scales, discussing which kinds of internal properties are really required for achieving them. In particular, we discuss ratio/difference representations, that have been studied to some extent, both theoretically and experimentally, but whose consequences have not been, as far as we know, fully exploited. We provide preliminary deterministic and probabilistic axiomatizations for finite structures and also mention some preliminary practical applications of these ideas.

Ratio scales have been a highly controversial issue between physicists and psychologists in the past (Rossi & Berglund, 2009). What can we say about them today? Perhaps we may first recall why they are so important. Campbell (1920/1957) notes that they enable measurement with a minimum degree of arbitrariness: once an arbitrary unit has been chosen the scale is entirely fixed. Stevens (1946) points out that statements concerning ratios, or percentages, are meaningful for them, whilst they are not for weaker scales. Finkelstein (2003) says that they are apt to represent rich relational structures, conformal to what he calls the paradigm of “strongly defined quantities”. But why are they controversial? The main question is whether it is possible or not to attain a ratio scale when a physical (or, more generally, empirical) addition operation is not present. Whilst in physics addition is often possible – it is possible to add lengths, masses, electrical resistances – this is not usually the case for perceptual characteristics. So this question is quite relevant for psychophysics, but also for physics, as we will discuss at a later stage. To answer this question, Campbell (1920) distinguished between fundamental and derived quantities: in his view, the former may be directly measured thanks to their internal properties, the latter may be measured only indirectly, thanks to some natural law linking them to some other measurable quantity (ies). Stevens (1946) circumvented this limitation by classifying measurement scales on the basis of their invariance with respect to group transformations. Yet he had to specify empirical relations related to each type of scale. For ratio scales he indicated “equality of ratios” as a distinctive feature, but he did not provide any axiomatization, in support of this statement. Successive axiomatizations of non additive ratio scales seem to have concerned magnitude estimation mainly, for which a few formal theories are presently available (Roberts, 1979; Narens, 1996). These theories, despite some differences, basically consider magnitude estimation as a kind of indirect measurement procedure, based on a cross-modality matching between the evoked sensation and some inner reference in persons. If this is the case, we should conclude that ratio scales may only be attained, directly, through an empirical addition operation, or, indirectly, by magnitude estimation.

Yet, in our opinion, there is another possible approach that leads to a *direct* definition of ratio scales, *without assuming physical addition*. This is based on the so called ratio/difference representations (Krantz et al., 1971; Miyamoto, 1983). These representations have been studied to some extent, both theoretically and experimentally, but their consequences have not been, as far as we know, fully exploited, which we will try and do in this paper.

Extensive versus Intensive Structures

The usual way for attaining a ratio scale is through an empirical *extensive structure*, that may be formally defined as a triple $\mathcal{S}_e = (A, \succsim, \circ)$, where A is a set of “objects” (events, persons) manifesting some characteristic, x , under consideration, \succsim is a weak order relation among objects and \circ is an (empirical) addition operation. A representation theorem for such a structure reads

$$a \sim b \circ c \Leftrightarrow m(a) = m(b) + m(c), \quad (1)$$

that is element a is equivalent to the empirical sum of b and c , if and only if the measure of a equals the sum of the measures of b and c . The associated scale is *ratio*, since the measure function m may safely undergo any similarity transformation

$$m' = \alpha m, \quad (2)$$

with $\alpha > 0$, which basically consists in a change of the measurement unit. Note that in such a structure there is not a “native”, so to speak, relation of ratio. Rather it is inferred by the ratio of measures, which is “meaningful”, since the scale is ratio. In other words, we say that the mass of a is twice the mass of b , if $m(a) = 2m(b)$.

Now a very good question is: *is it possible to attain a ratio scale through the internal properties (intra-relations) of the characteristic under examination, when they do not include empirical addition?* Or, in other words, are there empirical structures, having practical, technical and/or scientific relevance, that may give rise to a ratio scale, without having an empirical addition operation? If this is the case, such structures would provide a best representation of the measurement of the intensity of a sensation, for example, but also of, say, thermodynamic temperature, which is defined over a ratio scale, but does not seem to have any meaningful empirical addition operation. We will show in the following that such structures actually do exist, and we will call them *intensive structures*. For introducing them we have first to note that intervals may be compared and ordered in two ways, in respect of the *difference* of the extremes or in respect of their *ratio*. Interestingly enough, the underlying formal structures are the same! In other words, an interval scale may represent either a difference empirical system or a ratio empirical system. In order to understand how this may happen, consider that the key property for ensuring the representation is a monotonicity condition:

$$ab \sim a'b' \& bc \sim b'c' \Rightarrow ac \sim a'c', \quad (3)$$

which implies that the concatenation of adjacent intervals does not depend on where they are placed along a reference axis. But this may be true both for differences and for ratios! This is illustrated in figure 1.

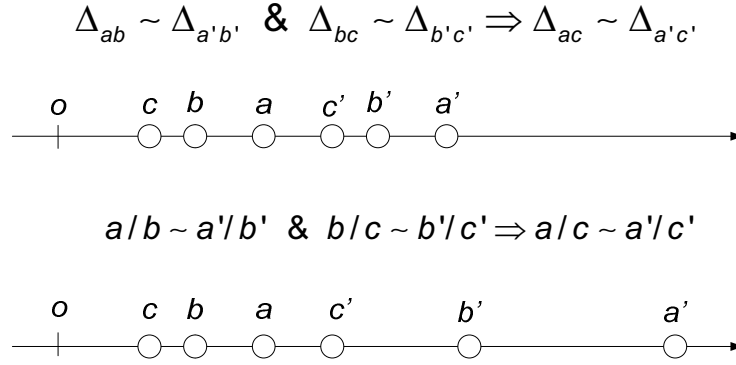


Fig. 1. Concatenation of intervals in respect of difference (upper part) and of ratios (lower part).

A good example of these two approaches is *spectrum measurement*. In that case it is possible to consider either a

- Constant-resolution spectrum, where spectral lines represent power associated to equal frequency intervals and the width of such intervals is a measure of the *resolution* of the analysis, or a
- Constant-relative-resolution spectrum, where spectral lines represent power associated to frequency intervals where the ratio of the extremes of each interval is constant and constitute a fraction of an octave.

So, as an alternative to extensive structures, we define an *intensive structure* as a triple $S_I = (A, \succ_d, \succ_r)$, where A is the usual set of objects and \succ_d and \succ_r are weak order relations among pairs of objects, referring, respectively, to *difference* and to *ratio*. Now if these two, distinct, orderings exist and if they satisfy some proper compatibility conditions, then it is possible to find a measure function, $m: A \rightarrow \mathbb{R}$, such that the following representations contemporarily holds true:

$$\Delta_{ab} \succ_d \Delta_{cd} \Leftrightarrow m(a) - m(b) \geq m(c) - m(d), \quad (3)$$

$$a/b \succ_r c/d \Leftrightarrow \frac{m(a)}{m(b)} \geq \frac{m(c)}{m(d)}, \quad (4)$$

where Δ_{uv} denotes the empirical difference between u and v , and u/v denotes their empirical ratio (not to be confused with the numerical ratio, here denoted by the horizontal line). It is possible to prove that such a measure is on a ratio scale, viz. it safely undergoes similarity transformations, $m' = \alpha m$, with $\alpha > 0$. We have no room here for discussing the associated mathematics, useful references may be found in Krantz et al. (1971/2007) and in Miyamoto (1983). Let us just briefly illustrate the meaning of such a result.

Suppose to present a set of stimuli to a group of persons, asking them to order them both in terms of differences and of ratios of the evoked sensations. If the results of the two orderings are compatible, in a sense that may be precisely specified, then a ratio scale may be applied. Actually this is the environment in which the above result has been obtained. Experimental work was performed, e.g., by Birnbaum (1980), Birnbaum and Veit (1974), Rule and Curtis (1980); early contributions by Garner (1954) and Torgerson are also noteworthy.

Interpretation related to physical measurements seems to be also possible. Consider indeed temperature measurement and the following thought experiment. Suppose that we have an un-calibrated constant-volume gas thermometer and an un-calibrated mercury-in-glass thermometer. We may use the former for comparing temperatures ratios and the latter for comparing differences. If the two resulting scales are in agreement, a ratio scale results. The underlying basic idea is that the matching is possible if a unique, absolute, zero point is identifiable.

Finite and Probabilistic Intensive Structures

So far we have suggested re-interpreting results concerning ratio/difference representations as a way for attaining a ratio scale by the internal properties of the characteristic under consideration, without needing an operation of empirical addition. This is what we have called an intensive structure. Moreover, since the original derivation of the representation theorem is quite cumbersome, a leaner one may be attained, by considering finite sets of objects. We have discussed elsewhere that this is not a real limitation and greatly simplifies the mathematics, rendering it understandable also to the non specialists. Furthermore, a probabilistic reformulation of representation results, necessary for expressing measurement uncertainty, requires, at the current state of the art, the hypothesis of finiteness (Rossi, 2006). Developing a finite theory does not seem to be that straightforward, since, in the case of finite structures it not possible to have, at the same time, a reference scale equally spaced for both differences and ratios. Actually this is one case in which the hypothesis of continuity seems to be really required! We have found two alternative axiomatizations, depending upon which equal spacing is maintained:

- A representation based on equal spacing of ratios is straightforward but rather counterintuitive: it corresponds to the previously examined case of constant-relative-resolution spectrum.
- One based on equally spacing for differences seems to require a stronger compatibility condition; in spite of that we think that it may be perhaps more appropriate.

In this latter case, the key condition may be stated in the following way.

Let $S = \{s_i, i = 1, \dots, n\}$ be a standard series for differences and let i, j, k be positive integers. Then we require that

$$a \sim s_i, b \sim s_j, a' \sim s_{k+i}, b' \sim s_{k+j} \Rightarrow a/b \sim a'/b', \quad (5)$$

where we have informally used the symbol “ \sim ” for denoting both equivalence between objects and equivalence between ratios. We have no room here for developing further the mathematical details; we plan to do that in a paper to be produced at a later stage.

Once a finite intensive structure has been defined, the notion of a *probabilistic intensive structure* may be introduced in a quite straightforward fashion. Let $\{S_i, i = 1, \dots, N\}$ be a finite collection of intensive structures on the same set of objects A , with an associated probability distribution \mathbb{P} . Then, the representation theorem now reads

$$\mathbb{P}(\Delta_{ab} \succ_d \Delta_{cd}) = P(m(a) - m(b) \geq m(c) - m(d)), \quad (6)$$

$$\mathbb{P}(a/b \succ_r c/d) = P\left(\frac{m(a)}{m(b)} \geq \frac{m(c)}{m(d)}\right), \quad (7)$$

A probabilistic representation allows accounting for the uncertainty that is inherent in any measurement process and that may be caused, in particular, by intra and inter-individual variability (Rossi, 2009).

Robust Magnitude Estimation

Lastly, we may briefly discuss some practical application of the above ideas. Consider, as an example, an experiment for the measurement of loudness. Suppose that a group of subjects are asked to rate sounds in terms of both loudness differences and loudness ratios and that $m', m'' : A \rightarrow \mathbb{R}$ are the corresponding resulting measure functions. If furthermore it is possible to fit data in such a way that, for each $a \in A$,

$$\begin{aligned} m(a) &\cong \alpha_1 (m'(a) + \beta), \\ m(a) &\cong \alpha_2 m''(a)^\gamma, \end{aligned} \tag{8}$$

then $m : A \rightarrow \mathbb{R}$ constitutes a measure function for loudness on a ratio scale. We will call such a procedure *robust magnitude estimation* (RME).

For implementing such a procedure we have developed a graphical interface shown in figure 2, for the two associated tasks of interval and ratio estimation.

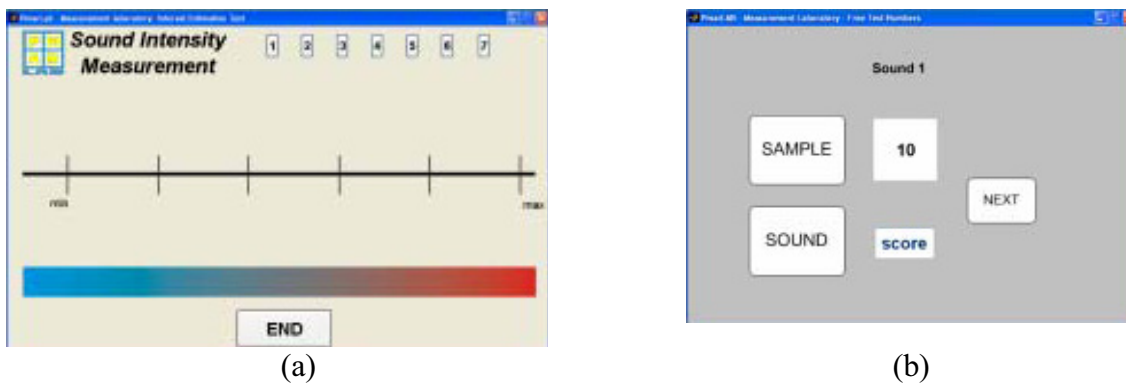


Fig. 2. Graphical interface for RME: a) Interval estimation; b) Ratio estimation.

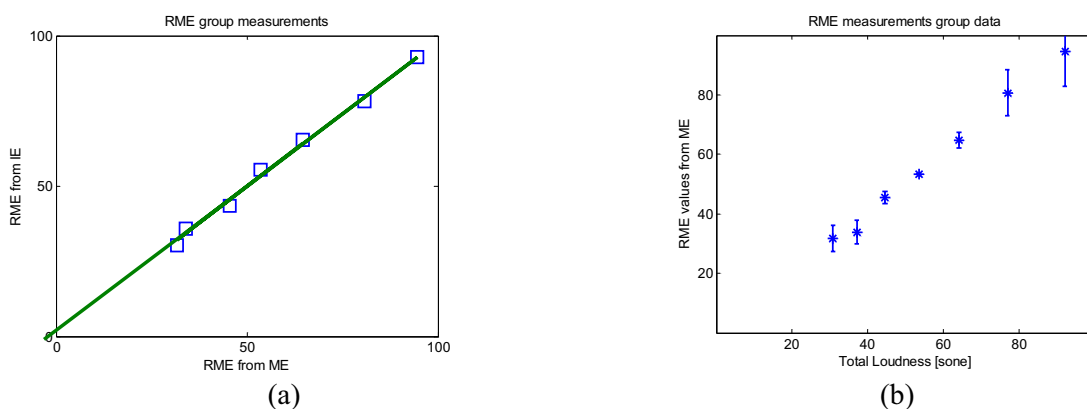


Fig. 3. a) Compatibility of results from magnitude estimation (abscissas) and interval estimation (ordinates); b) Comparison of results from Zwicker loudness (abscissas) and RME (ordinates).

In interval estimations, labels representing sounds are placed along a reference axis in such a way that their order and their distances represent the way they are perceived.

In ratio estimation, a reference sample is given the value of 10 and other sounds are given a number in agreement to their ratio with respect to the reference sound.

Such a procedure has been applied to the measurement of loudness of background noise in the working environment of operators in a port area (Crenna et al., 2008). Figure 3 presents some preliminary results, concerning: a) the agreement between the two perceptual scales, once that transformation (8) has been applied, and, b) the comparison with loudness measured in this way and Zwicker loudness.

Such results seem to indicate a good perspective for the practical application of this approach.

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