

WITH WHAT PROBABILITY REGULAR MINIMALITY CAN BE SATISFIED BY CHANCE?

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Abstract

A matrix of discrimination measures (e.g., probabilities) satisfies Regular Minimality (RM) if every row and every column of the matrix contains a single minimal entry, and an entry minimal in its row is minimal in its column. The probability with which a randomly chosen matrix complies with RM depends on how one defines “randomly chosen.” In this work we view all possible permutations of entries of a matrix without ties as equiprobable, and derive a closed-form expression for the probability with which a permutation yields a matrix satisfying RM.

Given a real-valued measure of discriminability $m(x,y)$ of stimuli $y \in Y$ from stimuli $x \in X$, *Regular Minimality (RM)* means that

- (A) for every $x \in X$ one can uniquely find a *matching stimulus* in Y , defined as the $y \in Y$ which is less discriminable from x than any other stimulus in Y ;
- (B) for every $y \in Y$ one can uniquely find a *matching stimulus* in X , defined as the $x \in X$ from which y is less discriminable than from any other stimulus in X ;
- (C) if y is the match for x in the sense (A), then x is the match for y in the sense (B).

The properties (A) and (B) should be qualified in terms of an appropriately defined equivalence relation, but we need not discuss this here. The RM principle was proposed in Dzhafarov (2002b) and further elaborated in Dzhafarov (2003), Dzhafarov and Colonius (2006a), and Kujala and Dzhafarov (2008, 2009). The notion of RM has nontrivial consequences for a variety of issues of traditional importance, ranging from Thurstonian-type modeling (see, e.g., Dzhafarov, 2006 in response to Ennis, 2006) to the “probability-distance” hypothesis (Dzhafarov, 2002a) to Fechnerian Scaling (see, e.g., Dzhafarov and Colonius, 2007) to matching-by-adjustment procedures (Dzhafarov & Perry, 2010) to the comparative version of the ancient “sorites” paradox (Dzhafarov & Dzhafarov, 2010, in press).

Regular Minimality for Rank-Order Matrices

In this paper we deal with the case when the stimulus sets X and Y are finite and the discrimination function $m(x,y)$ can be viewed as a matrix $M = \{m_{ij}\}$. This matrix must be square, $i, j \in \{1, \dots, n\}$, if it is to have a chance to comply with RM. We will assume that the matrix entries are pairwise distinct. Then the properties (A) and (B) are satisfied trivially (every row and every column has a unique minimal entry), and RM is reduced to the property (C):

an entry is minimal in its column if (hence also only if) it is minimal in its row.

The question we pose is this: how likely is it to obtain a matrix M satisfying this property “by chance”? One possible way to answer this question is to compute the proportion of the RM-compliant matrices among all matrices obtained from a fixed M by permuting its entries: if this proportion is very small, we conclude that RM, if observed, “could not have happened by chance alone.” This proportion is invariant with respect to all order-preserving transformations of m_{ij} ’s, the elements of M . We can therefore conveniently replace m_{ij} ’s with integers representing their ranks in the ordered sequence of all the entries: the smallest entry gets the rank 1, the next one the rank 2, etc., with the largest entry getting the rank n^2 . We will assume *no ties* among the entries and will refer to M with m_{ij} ’s replaced with their ranks as the *rank-order matrix* (corresponding to) M .

We can justify this approach by adopting the following “meta-probabilistic” view. Consider the entries of M not as data but as theoretical (population-level) values of a discriminability measure. Assuming that the possible values for m_{ij} form a set of reals I of a positive Lebesgue measure (e.g., an interval of reals, as in the case when m_{ij} are probabilities), we can assume that the entries are generated in accordance with some probability measure. Then we can translate the question of how likely it is to obtain an RM-compliant matrix “by chance” into the question of what the product measure is of the volume occupied by the RM-compliant matrices in I^{n^2} . If one and the same measure is imposed on all entries, then all permutations of any given set of entries are equiprobable. The absence of ties among the entries in this approach is ensured by additionally assuming that the probability measure imposed is absolutely continuous with respect to the Lebesgue measure. Then the product measure in question equals the proportion of the RM-compliant rank-order matrices among all possible rank-order matrices.

Probability of Obtaining RM by Chance

We begin by observing that if M satisfies RM, then so will any matrix M' obtained from M by an arbitrary permutation of its rows and/or columns. Indeed, the RM-compliance of M means the existence of bijections

$$h : \{1, \dots, n\} \rightarrow \{1, \dots, n\}, g : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$$

such that, for all $i, j \in \{1, \dots, n\}$, $m_{i, h(i)}$ is the minimal entry in the i 'th row, $m_{g(j), j}$ is the minimal entry in the j 'th column, and $g \equiv h^{-1}$. Following any permutations α and β of the rows and columns, respectively, m_{ij} of M is equal to $m_{\alpha(i), \beta(j)}$ of M' , whence the signs (and values) of the differences $m_{\alpha(i), \beta(j)} - m_{\alpha(i), \beta(h(i))}$ and $m_{\alpha(i), \beta(j)} - m_{\alpha(g(j)), \beta(j)}$ remain the same as those of $m_{ij} - m_{i, h(i)}$ and $m_{ij} - m_{g(j), j}$, respectively. This means that $m_{i, \beta(h(\alpha^{-1}(i)))}$ and $m_{\alpha(g(\beta^{-1}(j))), j}$ are the minima in the i 'th row and the j 'th column of M' , respectively, and it is clear that $\alpha \circ g \circ \beta^{-1} \equiv (\beta \circ h \circ \alpha^{-1})^{-1}$.

By appropriately chosen permutations α and β one can always bring any RM-compliant matrix M to a special form M' , in which the row and column minima are located on the main diagonal in the increasing order: $m'_{11} < \dots < m'_{nn}$. The procedure is illustrated on the 4×4 RM-compliant rank-order matrix below, where the first transformation, $M \rightarrow M^*$, is the permutations of rows $\alpha = \{1 \rightarrow 3, 2 \rightarrow 2, 3 \rightarrow 1, 4 \rightarrow 4\}$ and the second transformation, $M^* \rightarrow M'$, is the permutation of columns $\beta = \{1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 2, 4 \rightarrow 3\}$:

M	1	2	3	4	\implies	M^*	1	2	3	4	\implies	M'	1	2	3	4
1	12	14	4	3		1	1	11	13	15		1	1	13	15	11
2	8	10	2	9		2	8	10	2	9		2	8	2	9	10
3	1	11	13	15		3	12	14	4	3		3	12	4	3	14
4	16	5	6	7		4	16	5	6	7		4	16	6	7	5

Let us refer to RM-compliant rank-order matrices in this special form (minima on the diagonal in increasing order) as *special matrices*. Clearly, every RM-compliant rank-order matrix can be obtained from a special matrix by permuting its rows and columns, and as the number of all such permutations for an $n \times n$ matrix is $(n!)^2$, we have

$$R_n = (n!)^2 S_n, \quad (1)$$

where S_n is the number of all special matrices and R_n is the number of all RM-compliant rank-order matrices. As the number of all $n \times n$ rank-order matrices is $n^{2!}$, the probability of obtaining an RM-compliant matrix is given by

$$p_n = \frac{(n!)^2}{n^{2!}} S_n. \quad (2)$$

We turn now to computing the number of special matrices, S_n ($n \geq 2$). We need an auxiliary notion. For $k = 1, \dots, n$, we will call the set of cells

$$\{(i, k) : i < k\} \cup \{(k, j) : j < k\}$$

in an $n \times n$ matrix M the k th frame (this set is empty for $k = 1$). Clearly, M is the union of its diagonal entries and its frames. The letters f in the 4×4 matrix below indicate its frame cells and the dots fill the corresponding diagonal cells:

	1	2	3	4
1	•			
2				
3				
4				

	1	2	3	4
1		f		
2	f	•		
3				
4				

	1	2	3	4
1			f	
2			f	
3	f	f	•	
4				

	1	2	3	4
1				f
2				f
3				f
4	f	f	f	•

Let now diagonal entries (ranks) in an $n \times n$ rank-order matrix M have been chosen and arranged as $m_{11} = d_1 < \dots < m_{nn} = d_n$. We compute the number of ways in which we can fill the off-diagonal entries of M without violating the special form of RM ($m_{ij} > d_i$ and $m_{ij} > d_j$ if $i \neq j$). The n th frame should be filled by $2(n-1)$ ranks chosen from the set of $n^2 - d_n$ ranks exceeding d_n . The number of such choices is

$$P(n, d_n) = \begin{cases} \frac{(n^2 - d_n)!}{(n^2 - d_n - 2n - 2)!} & \text{if } n^2 - d_n \geq 2n - 2, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

None of these choices can violate the special form of RM, because any rank exceeding d_n will also exceed any d_k for $k < n$. For $k = 2, \dots, n-1$, let now all the frames above the k th have been filled without violating the special form of RM. The k th frame then should be filled by $2(k-1)$ numbers chosen from the set of $n^2 - d_k$ numbers exceeding d_k , from which however we should remove all the $n^2 - k^2$ numbers used up to fill in the previous $n - k$ frames and diagonal elements. That is, the k th frame can be filled in by $(n^2 - d_k) - (n^2 - k^2) = k^2 - d_k$ numbers taken $2(k-1)$ at a time. The number of such choices is

$$P(k, d_k) = \begin{cases} \frac{(k^2 - d_k)!}{(k^2 - d_k - 2k - 2)!} & \text{if } k^2 - d_k \geq 2k - 2, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

None of these choices can violate the special form of RM, because any rank exceeding d_k will also exceed any $d_{k'}$ for $k' < k$. Since, for any given n -tuple $\{d_1 < d_2 < \dots < d_n\}$ and any

$k = 2, \dots, n - 1$, the value of $P(k, d_k)$ does not depend on the fillings of the previous $n - k$ frames, the number of ways of filling all n frames of M is

$$\prod_{k=2}^n P(k, d_k). \quad (5)$$

Denoting the set of all n -tuples $\{d_1 < d_2 < \dots < d_n\}$ by D , the number of special matrices can be presented as

$$S_n = \sum_{\{d_1 < d_2 < \dots < d_n\} \in D} \prod_{k=2}^n P(k, d_k). \quad (6)$$

For computational purposes it is convenient to present the subset D^* of D for which the product in (6) is nonzero. Observe that for any n -tuple in D , $d_k \geq k$ because there are $k - 1$ diagonal entries less than d_k , and, by construction, $d_k < d_{k-1}$ if $k < n$. In view of (3) and (4) we should additionally require that $d_k \leq k^2 - 2(k - 1)$ for $k = 2, \dots, n$. It is easy to see, in particular, that the only possible values for the ranks d_2 and d_1 in D^* are 2 and 1, respectively. Presenting D^* in extenso,

$$S_n = \sum_{d_n=n}^{(n-1)^2-1} \dots \sum_{d_i=i}^{\min(d_{i-1}-1, (i-1)^2-1)} \dots \sum_{d_3=3}^{\min(d_4-1, 5)} \sum_{d_2=2}^2 \prod_{k=2}^n \frac{(k^2 - d_k)!}{(k^2 - d_k - 2k - 2)!}. \quad (7)$$

Using (2), the probability of an $n \times n$ matrix satisfying RM by chance is

$$p_n = \frac{(n!)^2}{n^2!} \sum_{\{d_1 < d_2 < \dots < d_n\} \in D} \prod_{k=2}^n \frac{(k^2 - d_k)!}{(k^2 - d_k - 2k - 2)!}. \quad (8)$$

We present the values of p_n for $n = 2, \dots, 10$:

n	p_n
2	0.33333333
3	0.10000000
4	0.0285714
5	0.0079365
6	0.0021645
7	0.0005828
8	0.0001554
9	0.0000411
10	0.0000108

This table shows that if one accepts our understanding of the notion “by chance” (based on considering all permutations of entries equiprobable), compliance with RM even for matrices as small as 4×4 or 5×5 should be taken seriously.

Concluding Remarks

The scope of the applicability of (8) is broader than it may seem. Note that x and y in the opening paragraph of this paper belong to different sets, X and Y . The main reason for this is that to avoid hopeless confusion when dealing with pairwise comparisons one should always keep in mind that the two stimuli being compared belong to distinct *observation areas* (see,

e.g., Dzhafarov & Colonius, 2006a). Even if x and y have the same value (say, they are line segments of the same length), they must occupy different spatial and/or temporal positions to be perceived as two distinct stimuli. So x and y should be designated as, say, $x = (5 \text{ cm, left})$ and $y = (5 \text{ cm, right})$, and with this rigorous designation X and Y cannot even overlap. We would like, however, to emphasize here that even the values of the elements of X and Y (ignoring the difference in the observation areas) need not be the same. Thus, in the probability matrix below RM is satisfied in the simplest form (the minima on the main diagonal) even though the values of the stimuli in the first observation area (rows) and in the second one (columns) are not the same:

	$y = 3.5$	$y = 5.5$	$y = 7.5$	$y = 11.5$
$x = 4.7$.25	.42	.64	.81
$x = 6.7$.51	.35	.55	.78
$x = 8.7$.63	.45	.14	.57
$x = 12.7$.76	.57	.34	.15

 \implies

	1	2	3	4
1	.25	.42	.64	.81
2	.51	.35	.55	.78
3	.63	.45	.14	.57
4	.76	.57	.34	.15

This observation makes it clear that even if a large matrix violates RM, one can look for its RM-compliant submatrices, as illustrated below:

	...	$y = 3.5$...	$y = 5.5$...	$y = 7.5$...	$y = 11.5$...
\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots
$x = 4.7$25426481	...
\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots
$x = 6.7$51355578	...
\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots
$x = 8.7$63451457	...
\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots
$x = 12.7$76573415	...
\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots	\vdots	\ddots

Further work is needed to derive the probability of obtaining by chance an RM-compliant submatrix in a matrix of a given size if one is allowed to “data-snoop,” that is, choose the submatrix after rather than prior to having studied the entire matrix.

Another line of work (suggested by Hans Colonius in a personal communication, December 2009) should lead to a formula for obtaining by chance an $n \times n$ matrix with a given number of violations of RM. A reasonable definition for the number of violations seems to be the number of rows (equivalently, columns) whose minima are not the minima of the columns (respectively, rows) in which they are found.

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