TESTING FOR REGULAR MINIMALITY

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Abstract

The law of regular minimality (RM) for same-different judgments was introduced by Dzhafarov as a fundamental property of discrimination and a necessary condition for Dzhafarov and Colonius' theory of Fechnerian scaling. A matrix of discrimination probabilities satisfies RM if every row and every column of the matrix contains a single minimal entry, and an entry minimal in its row is minimal in its column. In this paper we propose a measure based on which the compliance of a matrix of discrimination probabilities with RM can be tested statistically. The effectiveness of the measure for testing RM is demonstrated on real data.

Let $\Psi = (p_{ij})_{i,j=1,\ldots,n}$ be a square matrix of discrimination probabilities p_{ij} in the open unit interval $(0,1)$. The rows $i = 1,\ldots,n$ represent stimuli in one observation area (e.g., presented first or on the left), the columns $j = 1,\ldots,n$ represent stimuli in another observation area (presented second or on the right). An entry p_{ij} of the matrix denotes the probability with which the row and column stimuli i and j, when presented as an ordered pair (i,j), are judged to be different. The matrix Ψ satisfies *regular minimality* (RM) *in the canonical form* if any diagonal element p_{ii} is the single minimal entry in its row and column; that is, for every i, $p_{ii} < p_{il}$ for all $l \neq i$ and $p_{ii} < p_{ki}$ for all $k \neq i$. Any matrix obtained by permuting the columns of Ψ, where Ψ satisfies RM in the canonical form, is said to fulfill *RM in the* σ*-form*. In other words, any matrix $\Psi' = (p_{i\sigma(j)})_{i,j=1,\ldots,n}$, where $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a permutation and $(p_{ij})_{i,j=1,\ldots,n}$ satisfies RM in the canonical form, fulfills RM in the σ-form. An entry $p_{i\sigma(i)}$ then is the unique minimum entry in the ith row and in the $\sigma(i)$th column. RM in the σ-form (including the canonical form obtained for $\sigma = \mathrm{id}$ the identity) can be interpreted as: every stimulus, in either observation area, has a unique best match in the other observation area, and the relation "is the best match for" is symmetric.

RM as a fundamental principle of pairwise comparisons was proposed by Dzhafarov (2002b) and has been further elaborated in Dzhafarov (2003), Dzhafarov and Colonius (2006a), and Kujala and Dzhafarov (2008, 2009). This principle has nontrivial consequences for Thurstonian-type modeling (e.g., Dzhafarov, 2006; Ennis, 2006), the "probability-distance" hypothesis (Dzhafarov, 2002a), Fechnerian scaling (e.g., Dzhafarov & Colonius, 2006b, 2007), matching-by-adjustment procedures (Dzhafarov & Perry, 2010), and the comparative version of the ancient "sorites" paradox (Dzhafarov & Dzhafarov, 2010a, 2010b).

Deviation–Compliance Measure for Regular Minimality

Let $S_n = \{\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\} : \sigma \text{ a permutation}\}$ be the group of permutations on the set $\{1,\ldots,n\}$. In the following we propose a measure of deviation from or compliance with RM, when the form of RM is specified (fixed). An adaptation of this measure to accommodate the case when the form of RM is unspecified will also be briefly presented.

Let $\sigma \in S_n$ and $i \in \{1,\dots,n\}$. The *deviations* of Ψ from RM in the σ-form are given by $J_{r,\geq}(\Psi;\sigma,i) = \left\{ j \in \{1,\dots,n\} \setminus \{\sigma(i)\} : p_{i\sigma(i)} \geq p_{ij} \right\}$ (deviations in row i over columns) and $J_{c,\geq}(\Psi;\sigma,i) = \left\{ j \in \{1,\dots,n\} \setminus \{i\} : p_{i\sigma(i)} \geq p_{j\sigma(i)} \right\}$ (deviations in column $\sigma(i)$ over rows). The *compliances* of Ψ with RM in the σ-form are $J_{r,<}(\Psi;\sigma,i) = \left\{ j \in \{1,\dots,n\} : p_{i\sigma(i)} < p_{ij} \right\}$ (for compliances in row i over columns) and $J_{c,<}(\Psi;\sigma,i) = \left\{ j \in \{1,\dots,n\} : p_{i\sigma(i)} < p_{j\sigma(i)} \right\}$ (for compliances in column $\sigma(i)$ over rows). The following *deviation–compliance measure*, $m(\Psi;\sigma)$, for RM (in the σ-form) is proposed:

$$m(\Psi;\sigma) = \min_{1\leq i\leq n} m'(\Psi;\sigma,i), \tag{1}$$

where $m'(\Psi;\sigma,i) = \min\left(\min_{j\neq\sigma(i)}(p_{ij}) - p_{i\sigma(i)}, \min_{j\neq i}(p_{j\sigma(i)}) - p_{i\sigma(i)}\right)$.

This measure obviously satisfies the following properties.

- For all n, for all $\sigma \in S_n$, we have $-1 < m(\Psi;\sigma) < 1$.
- Ψ satisfies RM in the σ-form iff $m(\Psi;\sigma) > 0$. Equivalently, Ψ violates RM in the σ-form iff $m(\Psi;\sigma) \leq 0$.
- Let RM in the σ-form be satisfied. Then $m(\Psi;\sigma) > 0$ quantifies the "smallest" (worst) compliance of Ψ with RM in the σ-form: $m(\Psi;\sigma) = \min\left\{p_{ij} - p_{i\sigma(i)}, p_{j'\sigma(i')} - p_{i'\sigma(i')}\right\}$, where the minimum is taken over all $1 \leq i,i' \leq n$, $j \in J_{r,<}(\Psi;\sigma,i)$, and $j' \in J_{c,<}(\Psi;\sigma,i')$.
- Let RM in the σ-form be violated. Then $m(\Psi;\sigma) \leq 0$ quantifies the "largest" (worst) deviation of Ψ from RM in the σ-form: $m(\Psi;\sigma) = -\max\left\{p_{i\sigma(i)} - p_{ij}, p_{i'\sigma(i')} - p_{j'\sigma(i')}\right\}$, where the maximum is taken over all $1 \leq i,i' \leq n$, $j \in J_{r,\geq}(\Psi;\sigma,i)$, and $j' \in J_{c,\geq}(\Psi;\sigma,i')$.
- As a function $m(\Psi;\sigma) : (0,1)^{n^2} \to (-1,1)$ of the p_{ij}'s, $i,j = 1,\dots,n$, this measure is a continuous function on $(0,1)^{n^2}$.

Some remarks are in order with respect to the proposed measure. $m(\Psi;\sigma)$ allows for nuances of, both, violating or satisfying RM in a specified form. That is, it quantifies degrees of deviation from or compliance with RM. This can be useful information. For instance, a two-sample test problem for testing whether the degree of fulfilling RM in the σ-form in subpopulation 2 (e.g., females; $\Psi^{(2)}$) is larger than that in subpopulation 1 (e.g., males; $\Psi^{(1)}$), if RM in the σ-form is satisfied in both subpopulations, is $H_0 : m(\Psi^{(2)};\sigma) \leq m(\Psi^{(1)};\sigma)$ versus $H_1 : m(\Psi^{(2)};\sigma) > m(\Psi^{(1)};\sigma)$. The range of $m(\Psi;\sigma)$ is $(-1,1)$ and does not depend on n, the dimension of the stimulus space. This is good for using the measure descriptively. The measure only reflects deviation from or compliance with RM. In particular, degrees of deviation from or compliance with RM can be compared across different stimulus space dimensions. The measure $m(\Psi;\sigma)$ provides a characterization of RM in the σ-form: *Ψ satisfies RM in the σ-form iff $m(\Psi;\sigma) > 0$.* A test problem for RM in the σ-form can be proposed, $H_0 : m(\Psi;\sigma) > 0$ versus $H_1 : m(\Psi;\sigma) \leq 0$. Strictly speaking, this test problem is for rejecting RM in the σ-form. A test problem having RM in the σ-form in the alternative is $H_0 : m(\Psi;\sigma) \leq 0$ versus $H_1 : m(\Psi;\sigma) > 0$. In other words, the general idea underlying the present approach is to characterize RM in terms of a (range of) value(s) of some measure, and to test that measure for this (range of) value(s). The measure $m(\Psi;\sigma)$ depends on the permutation $\sigma \in S_n$. For instance, if $\sigma = \mathrm{id}$, then the previous tests are for RM in the canonical form. A deviation–compliance measure $m(\Psi)$ for RM, when the form of RM is unspecified, is

$$m(\Psi) = \max\left\{m(\Psi;\sigma) : \sigma \in S_n\right\}. \tag{2}$$

The following characterization holds: Ψ *satisfies RM in an unspecified form iff* $m(\Psi) > 0$. If $m(\Psi) > 0$, then there is exactly one permutation $\sigma_0 \in S_n$ with $\sigma_0 = \arg\max\{m(\Psi;\sigma) : \sigma \in S_n\}$, and the matrix Ψ satisfies RM in the σ_0-form and Ψ's degree of compliance with RM is quantified by $m(\Psi) = m(\Psi;\sigma_0)$. The test problems described for $m(\Psi;\sigma)$ analogously apply to the measure $m(\Psi)$; for instance, $H_0 : m(\Psi) > 0$ versus $H_1 : m(\Psi) \leq 0$. In the present paper, however, we only focus on the measure $m(\Psi;\sigma)$ and the test problem $H_0 : m(\Psi;\sigma) > 0$ versus $H_1 : m(\Psi;\sigma) \leq 0$.

In order to perform the test, the gradient of $m(\Psi;\sigma)$ must be computed (see the section "Maximum Likelihood Inference Methodology"). Call any entry $p_{i\sigma(i)}$ of the matrix Ψ a *pivot*. Assume that Ψ satisfies the following two conditions: (C1) There exists a *single* pivotal entry $p_{i_*\sigma(i_*)}$ such that $m'(\Psi;\sigma,i_*) = m(\Psi;\sigma)$. (C2) Moreover, there exists a *single* non-pivotal entry $p_{k_*l_*}$ among the i_*th row and $\sigma(i_*)$th column entries such that $p_{k_*l_*} - p_{i_*\sigma(i_*)} = m(\Psi;\sigma)$. It can be verified that under these conditions all partial derivatives of $m(\Psi;\sigma)$ with respect to the components of Ψ are constant:

$$\frac{\partial m(\Psi;\sigma)}{\partial p_{ij}} = \begin{cases} 1 & \text{if } (i,j) = (k_*,l_*), \\ -1 & \text{if } (i,j) = (i_*,\sigma(i_*)), \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We denote the set of all such matrices Ψ satisfying Conditions C1 and C2 by $\mathcal{D}_\sigma$.

Maximum Likelihood Inference Methodology

In this section we present an application of maximum likelihood (ML) asymptotic theory to discrimination probability matrices. For details on the theory of ML and related topics such as regularity conditions and asymptotic properties, see, for instance, Bishop, Fienberg, and Holland (1975), Casella and Berger (2002), and Lehmann and Casella (1998).

Let the population matrix Ψ be estimated by the matrix $\widehat{\Psi} = (\widehat{p_{ij}})_{i,j=1,\ldots,n}$ of relative frequencies $\widehat{p_{ij}} = n_{ij}/N$, where N is the sample size and n_{ij}'s are the success counts. We assume that the n_{ij}'s are the realizations of *independent* binomial distributions $B(N, p_{ij})$. The sample analogs of the population measures are obtained by replacing population values p_{ij} with sample values $\widehat{p_{ij}}$: $\widehat{m(\Psi;\sigma)} = m(\widehat{\Psi};\sigma)$ and $\widehat{m(\Psi)} = m(\widehat{\Psi})$. Given the fact that ML estimators for the binomial proportions are used, according to the invariance property of ML estimation, the sample measures are the ML estimators for the corresponding population values.

ML estimators possess a number of asymptotic properties under regularity conditions (see Casella & Berger, 2002, Section 10.6.2; Lehmann & Casella, 1998, Section 6.3). They are *asymptotically efficient* (the most precise estimates are produced), and implied by this property, they are *asymptotically normal*, *asymptotically unbiased* (the estimates converge in expectation to the true values), and *consistent* (the estimates converge in probability to the true values). A well-known fact is that the ML estimator for the binomial distribution, which belongs to the exponential family, fulfills required regularity conditions and is asymptotically efficient. The population measure $m(\Psi;\sigma)$ is a differentiable function in $\Psi \in \mathcal{D}_\sigma$. Therefore the sample measure $\widehat{m(\Psi;\sigma)}$ is an asymptotically efficient estimator for the population value $m(\Psi;\sigma)$, with $\Psi \in \mathcal{D}_\sigma$. Large sample normality with associated standard errors can be used to test hypotheses about this measure. In particular, a standard one-sided Z test can be applied to the test problem $H_0 : m(\Psi;\sigma) > 0$ versus $H_1 : m(\Psi;\sigma) \leq 0$. The null hypothesis is rejected to a significance level of α iff

$$\sqrt{N}\frac{\widehat{m(\Psi;\sigma)}}{\sigma_{\widehat{m(\Psi;\sigma)}}} < z_\alpha, \tag{4}$$

where $\widehat{\sigma_{\widehat{m(\Psi;\sigma)}}}$ is a consistent estimator for the population asymptotic standard error of the ML estimator $\widehat{m(\Psi;\sigma)}$ for $m(\Psi;\sigma)$, with $\Psi \in \mathcal{D}_\sigma$, and z_α denotes the lower α quantile of the unit normal distribution. This is the large sample significance test that we will apply to real data.

The exact asymptotic variance-covariance matrix of the ML estimator $\widehat{\Psi}$ for Ψ is given by the inverse of $\frac{1}{N}E_\Psi(-I)$, where I is the Hessian matrix of second-order partial derivatives of the log likelihood function, and $E_\Psi(-I)$, under regularity conditions, is the expected Fisher information matrix. This variance-covariance matrix is a diagonal matrix, $\operatorname{diag}(p_{ii}(1-p_{ii}))$, which has as (i,j)th element $p_{ii}(1-p_{ii})$ if $i=j$, and 0 if $i \neq j$. Using the delta method, the population asymptotic variance of the ML estimator $\widehat{m(\Psi;\sigma)}$ for $m(\Psi;\sigma)$ is given by

$$\sigma^2_{\widehat{m(\Psi;\sigma)}} = \nabla m(\Psi;\sigma)\cdot\left\{\left(\frac{1}{N}E_\Psi(-I)\right)^{-1}\cdot(\nabla m(\Psi;\sigma))^T\right\} \tag{5}$$

$$= p_{i_*\sigma(i_*)}\left(1-p_{i_*\sigma(i_*)}\right)+p_{k_*l_*}\left(1-p_{k_*l_*}\right), \tag{6}$$

where $\Psi \in \mathcal{D}_\sigma$ is the true parameter vector of binomial proportions and $\nabla m(\Psi;\sigma)$ is the gradient of $m(\Psi;\sigma)$ in Ψ. An estimator for $\sigma^2_{\widehat{m(\Psi;\sigma)}}$ is obtained by replacing Ψ with its ML estimator $\widehat{\Psi}$,

$$\widehat{\sigma^2_{\widehat{m(\Psi;\sigma)}}} = \nabla\widehat{m(\Psi;\sigma)}\cdot\left\{\operatorname{diag}\left(\widehat{p_{ii}}(1-\widehat{p_{ii}})\right)\cdot\left(\nabla\widehat{m(\Psi;\sigma)}\right)^T\right\} \tag{7}$$

$$= \widehat{p_{i_*\sigma(i_*)}}\left(1-\widehat{p_{i_*\sigma(i_*)}}\right)+\widehat{p_{k_*l_*}}\left(1-\widehat{p_{k_*l_*}}\right), \tag{8}$$

where $\nabla\widehat{m(\Psi;\sigma)} = \nabla m(\widehat{\Psi};\sigma)$ is the gradient of the measure $m(\Psi;\sigma)$ in $\widehat{\Psi}$. Note that the gradient in the data $\widehat{\Psi}$ may not necessarily exist. Note that $\widehat{\sigma^2_{\widehat{m(\Psi;\sigma)}}}$ may be zero. However, since the gradient in $\Psi \in \mathcal{D}_\sigma$ does exist, $p_{ij} \in (0,1)$ for $i,j = 1,\ldots,n$, and ML estimators satisfying such asymptotic properties as consistency are used, this is more unlikely as sample size increases.

Application to Real Data

Rothkopf's (1957) Morse code data of discrimination probabilities among 36 auditory Morse code signals for the letters $A,B,\ldots,Z$ and the digits $0,1,\ldots,9$ give the percentages each of roughly 150 subjects who responded "same" (choosing between "same" and "different") to the row signal followed by the column signal. The original Rothkopf's 36×36 data matrix $\widehat{\Psi_R}$ does not satisfy RM (in the canonical form). There are two minimal entries in the second row $\widehat{p_{2,2}} = \widehat{p_{2,24}} = 0.16$. Applying the approximate Z test yields $\sqrt{N}\frac{\widehat{m(\Psi_R;\sigma=\text{id})}}{\widehat{\sigma_{\widehat{m(\Psi_R;\sigma=\text{id})}}}} = \sqrt{150}\frac{0.16-0.16}{\sqrt{2\cdot 0.16\cdot(1-0.16)}} = 0$, which is a non-significant value of the test statistic at the levels of 0.05, 0.025, and 0.01. The corresponding p value is $\Phi\left(\sqrt{N}\frac{\widehat{m(\Psi_R;\sigma=\text{id})}}{\widehat{\sigma_{\widehat{m(\Psi_R;\sigma=\text{id})}}}}\right) = 0.5$, where Φ is the cumulative distribution function of the standard normal distribution. According to this test, the null hypothesis $H_0 : m(\Psi_R;\sigma=\text{id}) > 0$—that is, the population matrix Ψ_R underlying Rothkopf's data $\widehat{\Psi_R}$ fulfills RM—cannot be rejected.

Wish's (1967) Morse-code-like data of discrimination probabilities among 32 auditory Morse-code-like signals give the proportions each of roughly 108 subjects who responded "different" to the row signal followed by the column signal. The original Wish's 32×32 data matrix $\widehat{\Psi_W}$ does not satisfy RM (in the canonical form). There is the entry $\widehat{p_{20,22}} = 0.03$, which is the same as $\widehat{p_{22,22}}$ and smaller than $\widehat{p_{20,20}} = 0.06$. Applying our test to this data matrix $\widehat{\Psi_W}$

gives evidence that the population matrix Ψ_W underlying Wish's data $\widehat{\Psi_W}$ most likely satisfies RM. The value of the test statistic $\sqrt{N}\frac{\widehat{m(\Psi_W;\sigma=\text{id})}}{\sigma_{\widehat{m(\Psi_W;\sigma=\text{id})}}} = \sqrt{108}\frac{0.03-0.06}{\sqrt{0.03\cdot(1-0.03)+0.06\cdot(1-0.06)}} = -1.0662$ is non-significant at the levels of 0.05, 0.025, and 0.01. The corresponding p value is 0.1432. This p-value is smaller than the one obtained for Rothkopf's data.

Larger values of the measure $m(\Psi;\sigma)$ imply descriptively better degrees of RM. The degree of deviation of $\widehat{\Psi_W}$ from RM, as quantified by the measure's value, is higher than that of $\widehat{\Psi_R}$. We have $\widehat{m(\Psi_W;\sigma=\text{id})} = -0.03$, which is smaller than $\widehat{m(\Psi_R;\sigma=\text{id})} = 0$. Whether the degree of RM in the case of Wish's data is statistically significantly worse—that is, $H_0 : m(\Psi_W;\sigma=\text{id}) \geq m(\Psi_R;\sigma=\text{id})$ versus $H_1 : m(\Psi_W;\sigma=\text{id}) < m(\Psi_R;\sigma=\text{id})$—cannot be attained. The approximate standard two-sample one-sided Z test yields the test statistic value $\frac{\widehat{m(\Psi_W;\sigma=\text{id})}-\widehat{m(\Psi_R;\sigma=\text{id})}}{\sqrt{\left(\frac{\sigma_{\widehat{m(\Psi_W;\sigma=\text{id})}}}{\sqrt{108}}\right)^2+\left(\frac{\sigma_{\widehat{m(\Psi_R;\sigma=\text{id})}}}{\sqrt{150}}\right)^2}} = -0.5902$, which is non-significant at the levels of 0.05, 0.025, and 0.01. These results are in accordance with the results reported in Dzhafarov et al. (2010) on these data sets using a permutation test for RM. In spite of the violations Rothkopf's and Wish's data matrices can be deemed to provide compelling evidence for RM.

Discussion

We have proposed a measure based on which the fundamental property of RM for pairwise same-different comparisons can be tested statistically. The idea underlying the present approach is to characterize RM in terms of specific values of that measure and to test the measure for those values. A legitimate question is: Why not use the standard one-sided Z test with RM in the alternative, testing $H_0 : m(\Psi;\sigma) \leq 0$ versus $H_1 : m(\Psi;\sigma) > 0$? As extensive simulations have shown (not reported in this paper), the null hypothesis (RM being violated) is hardly ever rejected. The reason is that H_0 under this test is rejected if the test statistic $\sqrt{N}\frac{\widehat{m(\Psi;\sigma)}}{\sigma_{\widehat{m(\Psi;\sigma)}}} > z_{1-\alpha}$ exceeds the quantile $z_{1-\alpha}$. Now, once RM is violated in the data (but holds in population)—not seldom the case in practice—the test statistic becomes non-positive, and so the null hypothesis is not rejected. In the case of the two real data matrices $\widehat{\Psi_R}$ and $\widehat{\Psi_W}$, for instance, the test statistic values are zero and negative, respectively.

In-depth simulation studies have to be conducted to investigate the type I and type II errors of the proposed test, in finite sample sizes, over a broad range of population values of the measure and stimulus space dimensions. The presented approach has to be compared to the permutation test for RM by Dzhafarov et al. (2010). Other measures based on which RM (in specified or unspecified form) can be characterized and tested are possible. Measures can be proposed that quantify degrees of RM not only by the deviations or compliances involved, but also by the number of entries of the matrix violating or satisfying RM. They can trade off one against the other of these criteria. Nonparametric modifications of these measures are conceivable and statistically can deal with any dissimilarities (and not only with probabilities). This will be done in subsequent research on this topic, and we hope that this paper has shown the effectiveness of and has given motivation for such a measure for testing RM.

Acknowledgements

We are deeply grateful to Professor Ehtibar Dzhafarov for his invaluable comments on a first draft of the manuscript.

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