TESTING THE LOGISTIC PREDICTION OF THE RELATIVE JUDGMENT THEORY OF PSYCHOLOGICAL DISCRIMINATION

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ABSTRACT

The theory of comparative judgment proposed by Link (1975, 1992) received various tests and confirmations through the joint analysis of response times and response probabilities. These are strong tests of the general character of such a stochastic process. However this paper creates and applies a new approach to testing the prediction that the form of choice probabilities is logistic.

At the foundation of psychophysical theories of comparative judgment are assumptions regarding the characterizations of stimuli and how these are used to form a judgment. Fechner (1860, see Link 1994) represented stimuli by internal variables postulated to have Gaussian distributions because these distributions captured the 19th century idea about how error disturbed measurements, whether physical or mental.

The idea was sufficiently powerful to lead Fechner into a theory of comparative judgment that created the basis for experimental psychology (Link, 1994) as well as statistical hypothesis testing. The theory supposes that a particular internal stimulus value is compared to an internal threshold or criterion. Whether the internal stimulus value is above or below this threshold determines the choice response made by the subject. However, the theory was not able to account for the relations between response time and response probability, in part because the theory itself has nothing to do with the unfolding of a decision in time. What tests there are showed failure of the assumption that distance from the threshold or criterion determined response time. (e.g., Thomas and Myers, 1968).

The consideration of time-dependent mental processes begins with an entirely different view of the mechanism for creating a choice between two alternatives. In keeping with the general ideas introduced by Abraham Wald (1947) in the sequential analysis of statistical hypotheses Link introduced a distribution-free sequential theory of comparative judgment (Link and Heath, 1975). The theory's strength was in predicting relations between response time and response probability. Many tests showed the theory to provide an accurate portrayal of how subjects made judgments in many choice experiments.

Link (1978) showed that the response probabilities predicted by the theory had the form of a logistic function. This particular function was used by statisticians to fit data. But the source of the logistic equation was unknown. Its close fit to results in many different scientific areas suggested that there must be a common basis for its frequent appearance. The surprise was that this function describing choice probabilities was an outcome of the random walk theory of statistical hypothesis testing due to Wald (1947) and extended by Link (1978).

This logistic function depends on two parameters and defines the probability of choosing between one of two response alternatives as

$$P = \frac{1}{1+e^{-\theta A}} \qquad (1)$$

where θ is a measure of discriminability and A is the accumulated amount of comparative difference between the stimulus and a referent needed to trigger a response. The derivation and proof that this logistic function is a consequence of random walk theory is found in Link (1978).

The parameter θ that captures discriminability is derived from the formal analysis of the underlying stochastic accumulation of comparative differences through an application of Wald's Identity. Although Link (1978) derived the logistic response function as a consequence of bounded random walk theory, the parameter θ remained a discrimination parameter without formal relation to the particular nature of stimuli under judgment. In this sense the derivation did not depend on the form of the underlying probability distributions thought to characterize the stimuli.

A simple calculation shows that the unknown parameters θ and A are jointly determined by computing

$$\ln\left[\ln\left(\frac{P}{1-P}\right)\right] = ln\theta + lnA. \tag{2}$$

This addition of parameters, even on a logarithmic scale, suggests performing experiments designed to cause independent changes in θ and A in order to test whether the derived logistic form of the choice probabilities results from such experimental manipulations. The analysis of such an experiment is the focus of this paper.

The experiment occurred before these theoretical results were known. Link and Tindall (1971) and Link (1971) required subjects to compare sequentially presented horizontal line segments on a computer controlled display, and to judge whether a comparison line was the same or different from a fixed standard. Another change in performance resulted from requiring the subjects to respond under three different instructions regarding the speed and accuracy of their responses. Subjects were to respond as accurately as possible, to "Beat" a 460 msec response time deadline while being as accurate as possible, and to "Beat" a 260 msec response time deadline while being as accurate as possible.

Four well-practiced subjects made choice judgments in 60-trial blocks within which the size of the comparative difference remained fixed as did the instructions on speed and accuracy. Within each block the comparison stimulus was either the same or different from the 2cm standard on 50 % of the trials. The first ten trials, with 50% same and 50% different comparisons, were treated as practice and do not enter into the analysis below. Each subject used the same speed and accuracy conditions during a day in which eight blocks of trials totaling 8*50=400 test trials yielded 100 trials for each of four different sizes of comparative difference, .1cm, .2cm, .3cm and .4cm. Each speed-accuracy condition ran for four successive days. Thus each subject contributed 1600 judgments for each speed-accuracy condition consisting of 400 judgments for each level of stimulus difference. The total number of test trials is 19,200.

The judgments proved to be quite similar in the probabilities of a correct response regardless of whether the response was "Same" or "Different" from the standard 2cm line (cf, Link, 1992, pp 214-223). For this reason, and to keep with the results as reported by Link and Tindall (1971), the response probabilities presented below are for the probabilities of a correct response whether the judgment was "Same" or "Different." Each row of Table 1 corresponds to a speed-accuracy condition for which there are a total of 6400 observations. Each column corresponds to a fixed amount of stimulus difference and a total per column of 4800 observations. As might be expected various amounts of stimulus difference and the speed-accuracy conditions caused large changes in the response probabilities.

Table 1: Observed and Predicted Correct Response Proportions

	1cm		2cm		3cm		4cm	
Speed-Accuracy	Obs.	Pred.	Obs.	Pred.	Obs.	Pred.	Obs.	Pred.
ACCURACY	.804	.834	.919	.918	.971	.965	.988	.983
460 msec	.731	.734	.818	.807	.876	.877	.905	.916
SPEED	.609	.595	.630	.637	.676	.684	.714	.720

The question about the adequacy of the logistic equation to describe these performances is best answered by analysis of individual subject data, rather than an analysis of the average data provided in Table 1. The same calculations can be applied in either case. To illustrate the ideas the probabilities in Table 1 are used below. Then the method will be applied to all four subject's individual results.

The additivity of effects due to θ and A requires a new form of analysis. By computing $\ln[\ln(P/(1-P))]=\ln\theta + \ln(A)$ all response proportions within each cell are a consequence of a sum of values dependent upon θ and A. Presumably these parameters depend upon the amount of stimulus difference, θ, and the amount of comparative difference, A, required to create a response. Thus, for each of J columns there is a value of θ_j. However, noting that these values may be averaged across a row i, (i=1,…,I) yields an average for row i of,

$$\frac{1}{J}\sum_{j=1}^{J}[\ln(\theta_j) + ln(A_i)] = \ln(A_i) + \theta^* \tag{3}$$

where $\theta^* = \ln(\theta_1\theta_2 \ldots \theta_J)^{1/J}$ the logarithm of the geometric mean of the unknown θ values. Similarly, averages down a column produce,

$$\frac{1}{I}\sum_{i=1}^{I}[\ln(\theta_j) + ln(A_i)] = \ln(\theta_j) + A^* \tag{4}$$

where $A^* = \ln(A_1A_2 \ldots A_I)^{1/I}$ the logarithm of the geometric mean of the unknown values of A. The overall mean, M, equals θ* + A*. These ideas are shown in Table 2. Here are the various probabilities in Table 1 converted to ln(ln()) values with averages shown in the penultimate right-hand column and the next to the lowest row. The overall mean M = 0.335.

Table 2 ln(ln(P/(1-P)))

	1cm	2cm	3cm	4cm	Average	Estimate of
ACCURACY	0.345	0.887	1.256	1.484	0.993	$\theta^* + \ln A_{ACC}$
460	0.000	0.407	0.670	0.813	0.473	$\theta^* + \ln A_{460}$
260	-0.814	-0.631	-0.307	-0.089	-0.460	$\theta^* + \ln A_{260}$
Average	-0.157	0.221	0.540	0.736	0.335	$=M=\theta^*+A^*$
Estimate of	$\ln\theta_1 + A^*$	$\ln\theta_2 + A^*$	$\ln\theta_3 + A^*$	$\ln\theta_4 + A^*$		

Although all the parameter values are unknown, a test of the logistic equation is still possible. Note that the sum of averages for a particular row and column (i, j) gives

$$\begin{aligned}\theta^* + lnA_i + A^* + ln\theta_j &= lnA_i + ln\theta_j + \theta^* + A^* \\ &= lnA_i + ln\theta_j + M.\end{aligned} \tag{5}$$

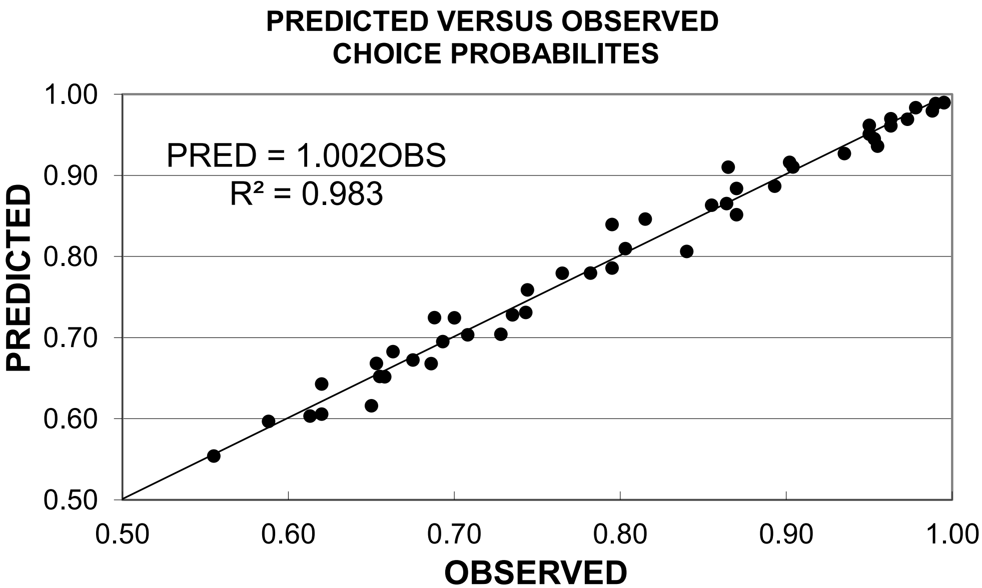

Figure 1. Response probabilities observed and predicted for the experiment of Link and Tindall (1971) employing speed-accuracy instructions and changes in discriminability.

Thus, by subtracting M from the sum of a row and column averages and exponentiating the result, the unknown value of $\theta_j A_i$ is estimated. These may then be used in Equation 1 to predict the response probabilities. If the logistic model is correct then there should be a close correspondence between the predicted and observed probabilities. Notice too that this is not trivial, the same value for θ is assumed to apply across the speed-accuracy conditions and a fixed value for A is assumed to apply across values of stimulus difference.

For the purpose of illustration Table 1 includes predicted values for these marginal response probabilities, and the fit is quite good. However, a better test is to apply the method to each subject separately and then examine the overall comparison between observed and predicted values. These values for the 12 probabilities for each of the four subjects appear in Figure 1. The linear equation of best fit with fixed zero intercept shows good agreement between observed and predicted response probabilities. The average deviation from predicted values is 0.002.

The good agreement between predicted and observed probabilities suggests carrying this analysis a step further. Notice that each row or column average may have the value M subtracted from it to leave only those parameters related to the experimental condition assigned to either the row or column. That is, for column 1:

$$\text{Average Column } 1 - M = A^* + \ln\theta_1 - (\theta^* + A^*) = \ln\theta_1 - \theta^*. \qquad (6)$$

This value equals the logarithm of θ_1 minus the logarithm of the geometric mean of the θs. Exponentiating this result yields the value of θ divided by the geometric mean of the theta values – a relative measure for θ.

These values for the average results and for averages of subject's relative parameters are shown in Figure 2. These are relative values of the discrimination parameter θ for each level of stimulus difference, .1cm, .2cm, .3cm and .4cm. Of course these values increase as the size of the stimulus difference increases.

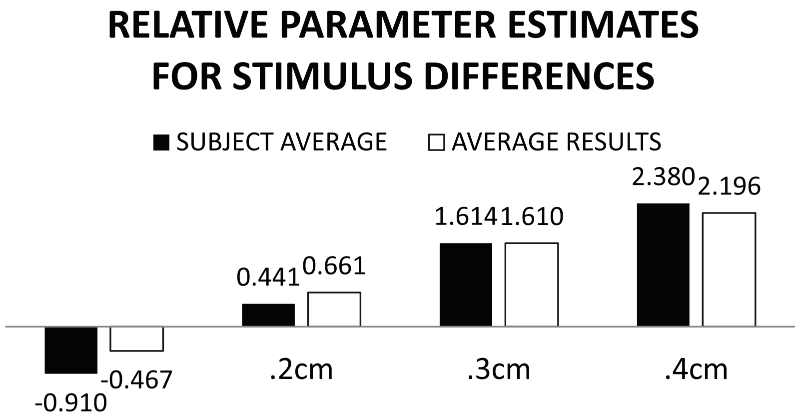

Figure 2. Relative estimates of the effects of stimulus difference on performance. The first column corresponds to .1cm. Averages of subject estimates agree closely with those obtained from the average results in Table 1.

The real question is how these values are related to the physical values of the stimuli being compared. The fact that this ratio is a dimensionless quantity suggests that the numerator and denominator be estimated by using the actual stimulus values.

There are no physical values for the parameter A, thought to change under changes in the speed-accuracy instructions given the subjects. The physical parameters corresponding to the instructions may be related to a subject's perception of how much stimulus difference must be accrued to make a response or in some way to the time to respond. But at this time the association with the relative speed-accuracy parameters shown in Figure 3 is unknown. The idea that the amount of accumulated stimulus difference needed to respond is the measure of the parameter A suggests that A should increase as the speed deadlines are relaxed, and this is quite clear in Figure 3.

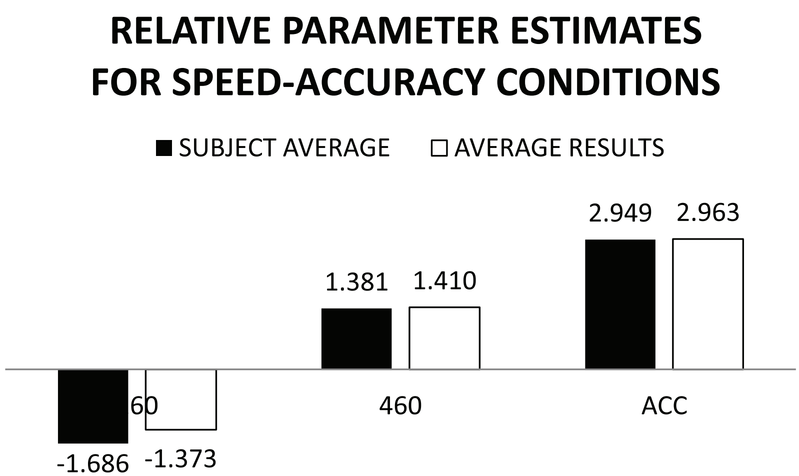

Figure 3. Relative values of the parameter A controlling the average duration of responses.

This next step toward understanding the relationship between the theoretical parameters and the physical values that underlie them requires a deeper analysis of the underlying probability distributions representing the stimuli and the particular mechanism for creating a comparative difference. The small amount of space available here does not allow for this extended discussion.

However, these tests of the predicted logistic representation of response proportions seems sufficiently strong to provide excellent support for the sequential theory of psychological discrimination first proposed in Link and Heath (1975).

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